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2006 J. Phys. A: Math. Gen. 39 7641

(http://iopscience.iop.org/0305-4470/39/24/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7641–7655 doi:10.1088/0305-4470/39/24/003

A note on the dispersionless BKP hierarchy

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Received 27 March 2006
Published 31 May 2006
Online at stacks.iop.org/JPhysA/39/7641

Abstract

We study the integrable hierarchy underlying topological Landau–Ginzburg models of *D*-type proposed by Takasaki. Since this integrable hierarchy contains the dBKP hierarchy as a sub-hierarchy, we refer it to the extended dBKP (EdBKP) hierarchy. We give a dressing formulation to the EdBKP hierarchy and investigate additional symmetries associated with the solution space of the hierarchy. We obtain hodograph solutions of its finite-dimensional reductions via Riemann–Hilbert problem (twistor construction) and derive Bäcklund transformations of the (2 + 1)-dimensional dBKP equation from additional flows. Finally, the modified partner of the dBKP hierarchy is also established through a Miura transformation.

PACS number: 02.30.Ik

1. Introduction

Dispersionless KP(dKP) hierarchy has been one of prototype systems in dispersionless integrable hierarchies, which plays an important role in theoretical and mathematical physics (see, e.g., [1, 3, 10–13, 15, 16, 24, 26] and references therein). A variant system of the dKP, the so-called dBKP hierarchy [21], is still at the early stage for studying integrability. The dBKP hierarchy can be considered as the dispersionless limit (or quasi-classical limit) of the BKP hierarchy (the KP of B-type) [9] which is a kind of reduction of the KP hierarchy. Such a reduction however is quite different from that for the KdV hierarchy. In the past few years, some progresses have been made for the dBKP hierarchy such as hodograph transformations [8], w-infinity symmetries [21] and $\bar{\partial}$-dressing method [6, 14]. In [22], Takasaki proposed an integrable hierarchy to study topological Landau–Ginzburg models of *D*-type. This new integrable hierarchy resembling to the dispersionless Toda (dToda) hierarchy [23, 25, 26] has two sets of time variables. Motivated by the Riemann–Hilbert approach to the dToda hierarchy [23, 26], we show that this new hierarchy enables us to investigate several properties associated with the dBKP hierarchy including dressing formulation, finite-dimensional reductions, hodograph solutions, additional symmetries and Bäcklund transformations. Our results not only give a supplement to the previous studies but also provide a more complete picture of the dBKP hierarchy.

Let us briefly recall the BKP hierarchy and its quasi-classical limit. The BKP hierarchy is defined by the Lax equations [9]

$$\partial_{2n+1} L = [B_{2n+1}, L], \qquad B_{2n+1} = (L^{2n+1})_+ \tag{1.1}$$

with constraint

$$L^* = -\partial L \partial^{-1} \tag{1.2}$$

where the Lax operator has the form

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots,$$

with coefficient functions u_i depending on the time variables $t = (t_1, t_3, \ldots)$ and $\left(\sum_i a_i \partial^i\right)_+ = \sum_{i\geqslant 0} a_i \partial^i$. It can be shown [9] that the constraint (1.2) is equivalent to the condition $(B_{2n+1})_{[0]} = 0$ with $\left(\sum_i a_i \partial^i\right)_{[j]} = a_j$. The Lax equation (1.1) can be described by the compatibility condition of the linear system

$$L\psi = \lambda\psi, \qquad \partial_{2n+1}\psi = B_{2n+1}\psi. \tag{1.3}$$

Let us take the dispersionless limit to the BKP. Under the change of variables $t \to t/\epsilon$ and assuming that $u_i(t/\epsilon) = u_i(t) + O(\epsilon)$ and $\psi = \exp(S/\epsilon)$, the linear system (1.3) in the limit $\epsilon \to 0$ gives $\lambda = k + \sum_{j=1}^{\infty} u_{i+1} k^{-i}$ and the phase function S satisfies

$$\partial_{2n+1} S = \mathcal{B}_{2n+1} \tag{1.4}$$

where $k = S_x$ and $\mathcal{B}_{2n+1} = (\lambda^{2n+1})_{\geqslant 0}$. From now on, the projections are with respect to Laurent series of k as $\left(\sum_i a_i k^i\right)_{\geqslant l} = \sum_{i\geqslant l} a_i k^i$, $\left(\sum_i a_i k^i\right)_{<l} = \sum_{i<l} a_i k^i$ and $\left(\sum_i a_i k^i\right)_{[l]} = a_l$. To incorporate the dispersionless limit of the constrained equation (1.2), we apply both sides of $L = -\partial^{-1} L^* \partial$ on $\psi = \exp(S/\epsilon)$. Then the left-hand side in the limit $\epsilon \to 0$ gives $\lambda = k + \sum_{i=1}^{\infty} u_{i+1} k^{-i}$, while for the right-hand side $\lambda = k + \sum_{i=1}^{\infty} (-1)^{i+1} u_{i+1} k^{-i}$. This means that $\lambda(t, -k) = -\lambda(t, k)$ or

$$\lambda = k + \sum_{i=1} u_{2i} k^{-2i+1} = k + u_2 k^{-1} + u_4 k^{-3} + \cdots.$$

Note that $\lambda(t, k)$ is an odd function in k, hence $\mathcal{B}_{2n+1}(t, -k) = -\mathcal{B}_{2n+1}(t, k)$ or $\mathcal{B}_{2n+1}(t, k) = k^{2n+1} + \sum_{j=1}^{n} b_{n,2j-1}(t) k^{2j-1}$. Differentiating equation (1.4) over x we have $\partial_{2n+1} k = \partial_x \mathcal{B}_{2n+1}$ which, after expressing in λ, is nothing but the Lax equation of the dBKP hierarchy [21]

$$\partial_{2n+1}\lambda = \{\mathcal{B}_{2n+1}, \lambda\} \tag{1.5}$$

where the Poisson bracket is defined by

$$\{f, g\} = \frac{\partial f}{\partial k}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial k}. \tag{1.6}$$

The simplest nontrivial flow in the dBKP hierarchy is the (2 + 1)-dimensional dBKP equation [8]:

$$3u_t + 15u^2 u_x - 5uu_y - 5u_x \partial_x^{-1} u_y - \tfrac{5}{3}\partial_x^{-1} u_{yy} = 0, \tag{1.7}$$

where $t_1 = x$, $t_3 = y$, $t_5 = t$ and $u \equiv u_2$.

2. Dressing formulation

Takasaki [22] proposed an integrable hierarchy underlying topological Landau–Ginzburg models of D-type as

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial t_{2n+1}} &= \{\mathcal{B}_{2n+1},\mathcal{L}\}, & \frac{\partial\mathcal{L}}{\partial \hat{t}_{2n+1}} &= \{\hat{\mathcal{B}}_{2n+1},\mathcal{L}\} \\ \frac{\partial\hat{\mathcal{L}}}{\partial t_{2n+1}} &= \{\mathcal{B}_{2n+1},\hat{\mathcal{L}}\}, & \frac{\partial\hat{\mathcal{L}}}{\partial \hat{t}_{2n+1}} &= \{\hat{\mathcal{B}}_{2n+1},\hat{\mathcal{L}}\}, \qquad n = 0,1,2,\ldots \end{aligned} \tag{2.1}$$

with

$$\mathcal{L} = k + \sum_{n=1}^{\infty} u_{2n}k^{-2n+1}, \qquad \hat{\mathcal{L}} = \sum_{n=0}^{\infty} \hat{u}_{2n}k^{2n+1}, \qquad \hat{u}_0 \neq 0$$

and

$$\mathcal{B}_{2n+1} = (\mathcal{L}^{2n+1})_{\geqslant 0}, \qquad \hat{\mathcal{B}}_{2n+1} = (\hat{\mathcal{L}}^{-2n-1})_{\leqslant -1},$$

where the coefficient functions u_{2n} and $\hat{u}_{2n}$ depend on the time variables $t = (t_1, t_3, \ldots)$ and $\hat{t} = (\hat{t}_1, \hat{t}_3, \ldots)$ and the Poisson bracket $\{\,,\}$ here is defined by (1.6). Since $\mathcal{L}(-k) = -\mathcal{L}(k)$ and $\hat{\mathcal{L}}(-k) = -\hat{\mathcal{L}}(k)$, we have $(\mathcal{B}_{2n+1})_{[0]} = (\hat{\mathcal{B}}_{2n+1})_{[0]} = 0$. The Lax equations (2.1) are equivalent to the zero curvature equations

$$\begin{aligned} &\frac{\partial\mathcal{B}_{2m+1}}{\partial t_{2n+1}} - \frac{\partial\mathcal{B}_{2n+1}}{\partial t_{2m+1}} + \{\mathcal{B}_{2m+1},\mathcal{B}_{2n+1}\} = 0, \\ &\frac{\partial\hat{\mathcal{B}}_{2m+1}}{\partial \hat{t}_{2n+1}} - \frac{\partial\hat{\mathcal{B}}_{2n+1}}{\partial \hat{t}_{2m+1}} + \{\hat{\mathcal{B}}_{2m+1},\hat{\mathcal{B}}_{2n+1}\} = 0, \\ &\frac{\partial\mathcal{B}_{2m+1}}{\partial \hat{t}_{2n+1}} - \frac{\partial\hat{\mathcal{B}}_{2n+1}}{\partial t_{2m+1}} + \{\mathcal{B}_{2m+1},\hat{\mathcal{B}}_{2n+1}\} = 0, \end{aligned} \tag{2.2}$$

which guarantees that the Lax equations (2.1) commute between themselves. Since the first equation of (2.1) (or (2.2)) involving t_{2n+1}-flows only is just the dBKP hierarchy (1.5), thus (2.1) (or (2.2)) is an integrable extension of the dBKP hierarchy by introducing an extra set of time variables $\hat{t}_{2n+1}$. We refer (2.1) (or (2.2)) to the extended dBKP (EdBKP) hierarchy.

Next, we like to show that the Lax operators of the EdBKP hierarchy have a dressing formulation, similar to that of the dToda case [26], as

$$\mathcal{L} = \mathrm{e}^{\mathrm{ad}\varphi(t,\hat{t},k)}(k), \qquad \hat{\mathcal{L}} = \mathrm{e}^{\mathrm{ad}\hat{\varphi}(t,\hat{t},k)}(k)$$

where $\mathrm{ad}\,X(Y) \equiv \{X, Y\}$ and the dressing functions $\varphi(t,\hat{t},k)$ and $\hat{\varphi}(t,\hat{t},k)$ are defined by

$$\varphi(t,\hat{t},k) = \sum_{n=1}^{\infty} \varphi_{2n}(t,\hat{t})k^{-2n+1}, \qquad \hat{\varphi}(t,\hat{t},k) = \sum_{n=1}^{\infty} \hat{\varphi}_{2n}(t,\hat{t})k^{2n-1}.$$

Then the Lax equations (2.1) imply that

$$\begin{aligned} \nabla_{t_{2n+1},\varphi}\varphi &= -(\mathcal{L}^{2n+1})_{\leqslant -1}, & \nabla_{\hat{t}_{2n+1},\varphi}\varphi &= (\hat{\mathcal{L}}^{-2n-1})_{\leqslant -1} \\ \nabla_{t_{2n+1},\hat{\varphi}}\hat{\varphi} &= (\mathcal{L}^{2n+1})_{\geqslant 1}, & \nabla_{\hat{t}_{2n+1},\hat{\varphi}}\hat{\varphi} &= -(\hat{\mathcal{L}}^{-2n-1})_{\geqslant 1} \end{aligned} \tag{2.3}$$

where $\nabla_{t_n,X}Y = \sum_{k=0}^{\infty}(\mathrm{ad}\,X)^k\partial_{t_n}Y/(k+1)!$.

In order to discuss Riemann–Hilbert problem, let us introduce the Orlov–Schulman operators [20] through the dressing operator approach as

$$\mathcal{M} = \mathrm{e}^{\mathrm{ad}\varphi(t,\hat{t},k)}\left(\sum_{n=1}^{\infty}(2n+1)t_{2n+1}k^{2n} + x\right) = \sum_{n=0}^{\infty}(2n+1)t_{2n+1}\mathcal{L}^{2n} + \sum_{n=0}^{\infty} v_{2n+2}\mathcal{L}^{-2n-2},$$

$$\hat{\mathcal{M}} = \mathrm{e}^{\mathrm{ad}\hat{\varphi}(t,\hat{t},k)}\left(-\sum_{n=0}^{\infty}(2n+1)\hat{t}_{2n+1}k^{-2n-2} + x\right) = -\sum_{n=0}^{\infty}(2n+1)\hat{t}_{2n+1}\hat{\mathcal{L}}^{-2n-2} + \sum_{n=0}\hat{v}_{2n+2}\hat{\mathcal{L}}^{2n}$$

which, by (2.3), satisfy the Lax equations

$$\begin{aligned}\frac{\partial\mathcal{M}}{\partial t_{2n+1}} &= \{\mathcal{B}_{2n+1},\mathcal{M}\}, \qquad \frac{\partial\mathcal{M}}{\partial\hat{t}_{2n+1}} = \{\hat{\mathcal{B}}_{2n+1},\mathcal{M}\}\\ \frac{\partial\hat{\mathcal{M}}}{\partial t_{2n+1}} &= \{\mathcal{B}_{2n+1},\hat{\mathcal{M}}\}, \qquad \frac{\partial\hat{\mathcal{M}}}{\partial\hat{t}_{2n+1}} = \{\hat{\mathcal{B}}_{2n+1},\hat{\mathcal{M}}\}, \qquad n = 0,1,2,\ldots\end{aligned} \tag{2.4}$$

and the canonical Poisson relation

$$\{\mathcal{L},\mathcal{M}\} = \{\hat{\mathcal{L}},\hat{\mathcal{M}}\} = 1. \tag{2.5}$$

Note that $\mathcal{M}$ and $\hat{\mathcal{M}}$ are even functions in k, i.e. $\mathcal{M}(-k) = \mathcal{M}(k)$ and $\hat{\mathcal{M}}(-k) = \hat{\mathcal{M}}(k)$. The integrability of the hierarchy can be viewed from the canonical conjugate pair $(\mathcal{L},\mathcal{M})$ and $(\hat{\mathcal{L}},\hat{\mathcal{M}})$ which provides the Darboux coordinates of the 2-form $\omega = \sum_{n=0}^{\infty}\mathrm{d}\mathcal{B}_{2n+1}\wedge\mathrm{d}t_{2n+1} + \sum_{n=0}^{\infty}\mathrm{d}\hat{\mathcal{B}}_{2n+1}\wedge\mathrm{d}\hat{t}_{2n+1}$ (with $\mathrm{d}\omega = 0$ and $\omega\wedge\omega = 0$) so that

$$\omega = \mathrm{d}\mathcal{L}\wedge\mathrm{d}\mathcal{M} = \mathrm{d}\hat{\mathcal{L}}\wedge\mathrm{d}\hat{\mathcal{M}}$$

imples (2.1), (2.4) and (2.5). It can be shown [26] that there exists a single function $\mathcal{F}(t,\hat{t})$ called free energy such that the coefficients $v_{2n+2}(\hat{v}_{2n+2})$ in $\mathcal{M}(\hat{\mathcal{M}})$ and those $f_{2n}(\hat{f}_{2n})$ in the inverse function $k(\mathcal{L},t,\hat{t})(k(\hat{\mathcal{L}},t,\hat{t}))$,

$$k(\mathcal{L}) = \mathcal{L} - \sum_{n=1} f_{2n}\mathcal{L}^{-2n+1} = \sum_{n=0}\hat{f}_{2n}\hat{\mathcal{L}}^{2n+1},$$

can be expressed in terms of second derivatives of the free energy as

$$\begin{aligned} f_{2n} &= \frac{1}{2n-1}\frac{\partial^2\mathcal{F}}{\partial x\partial t_{2n-1}}, \qquad & v_{2n+2} &= \frac{\partial\mathcal{F}}{\partial t_{2n+1}},\\ \hat{f}_{2n} &= -\frac{1}{2n+1}\frac{\partial^2\mathcal{F}}{\partial x\partial\hat{t}_{2n+1}}, \qquad & \hat{v}_{2n+2} &= -\frac{\partial\mathcal{F}}{\partial\hat{t}_{2n+1}},\end{aligned}$$

where $n \geqslant 1$ and $u = f_2 = \partial^2\mathcal{F}/\partial x^2$.

3. Additional symmetries

The solution space of the EdBKP hierarchy can be characterized by a Riemann–Hilbert problem (or twistor construction) [22]. Let $f(k,x)$, $g(k,x)$, $\hat{f}(k,x)$ and $\hat{g}(k,x)$ be functions with definite parity

$$\begin{aligned} f(-k,x) &= -f(k,x), \qquad & g(-k,x) &= g(k,x),\\ \hat{f}(-k,x) &= -\hat{f}(k,x), \qquad & \hat{g}(-k,x) &= \hat{g}(k,x),\end{aligned}$$

and satisfy the Poisson relations

$$\{f,g\} = \{\hat{f},\hat{g}\} = 1. \tag{3.1}$$

Then, the functional relations

$$f(\mathcal{L},\mathcal{M}) = \hat{f}(\hat{\mathcal{L}},\hat{\mathcal{M}}), \qquad g(\mathcal{L},\mathcal{M}) = \hat{g}(\hat{\mathcal{L}},\hat{\mathcal{M}})$$

imply the Lax equations and the canonical relation for $(\mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}})$. We call $(f, g, \hat{f}, \hat{g})$ the twistor data of the system. We remark that, similar to the dToda case [26], the dressing approach not only provides a convenient way to introduce the Orlov–Schulman operator but also enables us to prove the existence of the twistor data. The twistor data of the Riemann–Hilbert problem are by no means unique, instead there are infinitely many choices of them. They are connected to each other via canonical transformations. This brings out the notion of additional symmetries in the solution space. Let $F(k, x)$, $\hat{F}(k, x)$ be generating functions of the canonical transformation

$$\begin{aligned} &(f(k,x), g(k,x)) \to (f_\epsilon(k,x), g_\epsilon(k,x)) = \mathrm{e}^{-\epsilon \operatorname{ad} F}(f, g), \\ &(\hat{f}(k,x), \hat{g}(k,x)) \to (\hat{f}_\epsilon(k,x), \hat{g}_\epsilon(k,x)) = \mathrm{e}^{-\epsilon \operatorname{ad} \hat{F}}(\hat{f}, \hat{g}) \end{aligned} \tag{3.2}$$

where ϵ is an infinitesimal parameter and

$$\operatorname{ad} F = \frac{\partial F}{\partial k}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial k}, \qquad \operatorname{ad} \hat{F} = \frac{\partial \hat{F}}{\partial k}\frac{\partial}{\partial x} - \frac{\partial \hat{F}}{\partial x}\frac{\partial}{\partial k}.$$

Denoting

$$\mathcal{K}_\epsilon = \mathcal{K} + \epsilon \delta_{F,\hat{F}} \mathcal{K} + O(\epsilon^2),$$

where $\mathcal{K} = (\mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}}, \varphi, \hat{\varphi})$ and the derivative $\delta_{F,\hat{F}}$ has no effect on the time variables: $\delta_{F,\hat{F}} t_{2n+1} = \delta_{F,\hat{F}} \hat{t}_{2n+1} = 0$. Due to the parity of $(f, g, \hat{f}, \hat{g})$, we have

$$F(-k, x) = -F(k, x), \qquad \hat{F}(-k, x) = -\hat{F}(k, x),$$

and hence

$$F(k, x) = \sum_{ij} c_{ij} k^{2i+1} x^j, \qquad \hat{F}(k, x) = \sum_{ij} \hat{c}_{ij} k^{2i+1} x^j.$$

From (3.1) and (3.2) one can show that

$$\begin{aligned} \nabla_{\delta_{F,\hat{F}}\varphi} \varphi &= [F(\mathcal{L}, \mathcal{M}) - \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}})]_{\leqslant -1}, \\ \nabla_{\delta_{F,\hat{F}}\hat{\varphi}} \hat{\varphi} &= [\hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) - F(\mathcal{L}, \mathcal{M})]_{\geqslant 1}, \end{aligned}$$

where $\nabla_{\delta_{F,\hat{F}} X} Y$ is defined as before by replacing $\partial/\partial t_n$ by $\delta_{F,\hat{F}}$. Thus, from the dressing formulation the associated infinitesimal symmetries of $(\mathcal{L}, \hat{\mathcal{L}}, \mathcal{M}, \hat{\mathcal{M}})$ are given by

$$\begin{aligned} &\delta_{F,\hat{F}} \mathcal{L} = \{(F(\mathcal{L}, \mathcal{M}) - \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}))_{\leqslant -1}, \mathcal{L}\}, \quad \delta_{F,\hat{F}} \mathcal{M} = \{(F(\mathcal{L}, \mathcal{M}) - \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}))_{\leqslant -1}, \mathcal{M}\}, \\ &\delta_{F,\hat{F}} \hat{\mathcal{L}} = \{(\hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) - F(\mathcal{L}, \mathcal{M}))_{\geqslant 1}, \hat{\mathcal{L}}\}, \quad \delta_{F,\hat{F}} \hat{\mathcal{M}} = \{(\hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) - F(\mathcal{L}, \mathcal{M}))_{\geqslant 1}, \hat{\mathcal{M}}\}. \end{aligned} \tag{3.3}$$

From (2.1), (2.4) and (2.5), it is easy to show that the additional flows commute with the Lax flows $[\delta_{F,\hat{F}}, \partial_{t_{2n+1}}] = [\delta_{F,\hat{F}}, \partial_{\hat{t}_{2n+1}}] = 0$. However, in general, the additional flows do not commute between themselves. Given two pairs of generating functions $(F, \hat{F})$ and $(G, \hat{G})$, one can show that the infinitesimal symmetries $\delta_{F,\hat{F}}$ and $\delta_{G,\hat{G}}$ obey the commutation relations

$$[\delta_{F,\hat{F}}, \delta_{G,\hat{G}}] \mathcal{K} = \delta_{\{F,G\},\{\hat{F},\hat{G}\}} \mathcal{K}. \tag{3.4}$$

In fact, the commutation relation (3.4) can be decomposed into the following ones:

$$\begin{aligned} &[\delta_{F,0}, \delta_{G,0}] \mathcal{K} = \delta_{\{F,G\},0} \mathcal{K}, \\ &[\delta_{0,\hat{F}}, \delta_{0,\hat{G}}] \mathcal{K} = \delta_{0,\{\hat{F},\hat{G}\}} \mathcal{K}, \\ &[\delta_{F,0}, \delta_{0,\hat{G}}] \mathcal{K} = 0 \end{aligned}$$

which is isomorphic to a direct sum of w_∞-algebra and can be realized as infinite-dimensional Lie algebra of Poisson brackets on a two-dimensional phase space parameterized by (k, x) [2]. If one extends the infinitesimal symmetries to the free energy $\mathcal{F}$, then

$$\delta_{F,\hat{F}}\mathcal{F} = -\mathrm{res}[F^x(\mathcal{L},\mathcal{M})d_k\mathcal{L}] + \mathrm{res}[\hat{F}^x(\hat{\mathcal{L}},\hat{\mathcal{M}})d_k\hat{\mathcal{L}}]$$

where

$$F^x(k,x) = \int_0^x F(k,y)\,\mathrm{d}y, \qquad \hat{F}^x(k,x) = \int_0^x \hat{F}(k,y)\,\mathrm{d}y.$$

Particularly, the commutation relations for $\delta_{F,\hat{F}}$ on $\mathcal{F}$ receive a central extension term as

$$[\delta_{F,\hat{F}}, \delta_{G,\hat{G}}]\mathcal{F} = \delta_{\{F,G\},\{\hat{F},\hat{G}\}}\mathcal{F} + c(F,G) + \hat{c}(\hat{F},\hat{G}),$$

which can be decomposed into a direct sum of $w_{1+\infty}$-algebra:

$$[\delta_{F,0}, \delta_{G,0}]\mathcal{F} = \delta_{\{F,G\},0}\mathcal{F} + c(F,G),$$
$$[\delta_{0,\hat{F}}, \delta_{0,\hat{G}}]\mathcal{F} = \delta_{0,\{\hat{F},\hat{G}\}}\mathcal{F} + \hat{c}(\hat{F},\hat{G}),$$
$$[\delta_{F,0}, \delta_{0,\hat{G}}]\mathcal{F} = 0$$

where c and $\hat{c}$ are cocycles of the $w_{1+\infty}$-algebra defined by

$$c(F,G) = -\mathrm{res}[G(k,0)d_kF(k,0)], \qquad \hat{c}(\hat{F},\hat{G}) = \mathrm{res}[\hat{G}(k,0)d_k\hat{F}(k,0)]$$

4. Finite-dimensional reductions

In this section, we will discuss finite-dimensional reductions of the EdBKP hierarchy. Let us consider the following twistor data $(f, g, \hat{f}, \hat{g})$:

$$f(k,x) = k^m, \qquad g(k,x) = \frac{xk^{1-m}}{m} + h(k),$$
$$\hat{f}(k,x) = k^{m-2N}, \qquad \hat{g}(k,x) = \frac{xk^{2N-m+1}}{m-2N} + \hat{h}(k), \qquad m \in \text{odd}$$

which satisfy the canonical commutation relation (3.1), and the deformations $h(k)$ and $\hat{h}(k)$ are arbitrary even functions of k. The condition $f(\mathcal{L},\mathcal{M}) = \hat{f}(\hat{\mathcal{L}},\hat{\mathcal{M}})$ gives an N reduction of the EdBKP hierarchy defined by the Lax operator

$$\begin{aligned} L &= \mathcal{L}^m = \hat{\mathcal{L}}^{m-2N} \\ &= k^m + mu_2k^{m-2} + \left(mu_4 + \frac{m(m-1)}{2}u_2^2\right)k^{m-4} + \cdots + u_{2N}k^{m-2N}, \qquad u_{2N} = \hat{u}_0^{m-2N} \end{aligned}$$

which obeys the Lax equations

$$\partial_{t_{2n+1}}L = \{(\mathcal{L}^{2n+1})_{\geqslant 0}, L\}, \qquad \partial_{\hat{t}_{2n+1}}L = \{(\hat{\mathcal{L}}^{-2n-1})_{\leqslant -1}, L\}, \qquad n = 0, 1, 2, \ldots.$$

On the other hand, from the condition $g(\mathcal{L},\mathcal{M}) = \hat{g}(\hat{\mathcal{L}},\hat{\mathcal{M}})$, the projection $(\,)_{\leqslant 2N-m}$ implies

$$\begin{aligned} &\sum_{j=0}^{N-1}\frac{2j+1}{m}t_{2j+1}\mathcal{L}^{2j-m+1} + \sum_{j=N}^{\infty}\frac{2j+1}{m}t_{2j+1}\mathcal{L}^{2j-m+1}_{\leqslant 2N-m} + \sum_{j=0}\frac{v_{2j+2}}{m}\mathcal{L}^{-2j-m-1} + h(\mathcal{L})_{\leqslant 2N-m} \\ &\qquad = -\sum_{j=0}^{\infty}\frac{2j+1}{m-2N}\hat{t}_{2j+1}\hat{\mathcal{L}}^{2N-2j-m-1}_{\leqslant 2N-m} + \hat{h}(\hat{\mathcal{L}})_{\leqslant 2N-m}. \end{aligned}$$

Using $\mathrm{res}(\mathcal{L}^n d\mathcal{L}) = \delta_{n,-1}$, we obtain the hodograph equations

$$t^0_{2n+1}(u_j) = t_{2n+1} + \sum_{j=0}^{\infty} \mu^{(2n+1)}_{2j+1}(u_j) t_{2N+2j+1} + \sum_{j=0}^{\infty} \hat{\mu}^{(2n+1)}_{2j+1}(u_j)\hat{t}_{2j+1}, \qquad 0 \leqslant n \leqslant N-1, \tag{4.1}$$

where the characteristic speeds $\mu^{(2n+1)}_{2j+1}(u_j)$, $\hat{\mu}^{(2n+1)}_{2j+1}(u_j)$ and the initial positions $t^0_{2n+1}(u_j)$ are defined by

$$\mu^{(2n+1)}_{2j+1}(u_j) = \frac{2N+2j+1}{2n+1}\,\mathrm{res}\big(\mathcal{L}^{m-2n-2}\mathcal{L}^{2N+2j-m+1}_{\leqslant 2N-m} d_k\mathcal{L}\big), \tag{4.2}$$

$$\hat{\mu}^{(2n+1)}_{2j+1}(u_j) = \frac{m(2j+1)}{(m-2N)(2n+1)}\,\mathrm{res}\big(\mathcal{L}^{m-2n-2}\hat{\mathcal{L}}^{2N-2j-m-1}_{\leqslant 2N-m} d_k\mathcal{L}\big), \tag{4.3}$$

$$t^0_{2n+1}(u_j) = -\frac{m}{2n+1}\,\mathrm{res}\big(\mathcal{L}^{m-2n-2}(h(\mathcal{L}) - \hat{h}(\hat{\mathcal{L}}))_{\leqslant 2N-m} d_k\mathcal{L}\big). \tag{4.4}$$

To solve the associated hodograph solutions, we consider a class of initial positions. Since $t^0_{2n+1}(u_j)$ are determined by the deformation functions $(h, \hat{h})$ and thus the ambiguity of t^0_{2n+1} comes from the choice of $(h, \hat{h})$. Let $h(k)$ $(\hat{h}(k))$ be an arbitrary even function in k with Laurent series of the form $h(k) = \sum_i h_{2i}k^{2i}$ $\big(\hat{h}(k) = \sum_i \hat{h}_{2i}k^{2i}\big)$ where h_{2i} $(\hat{h}_{2i})$ are constants. Then, $t^0_{2n+1}(u_j)$ defined in (4.4) now can be expressed in terms of h_i, $\hat{h}_i$ and $\mu^{(2n+1)}_{2j+1}(u_j)$, $\hat{\mu}^{(2n+1)}_{2j+1}(u_j)$ as

$$t^0_{2n+1}(u_j) = -C_{2n+1} - \sum_{j\geqslant 0} C_{2N+2j+1}\mu^{(2n+1)}_{2j+1} - \sum_{j\geqslant 0} \hat{C}_{2j+1}\hat{\mu}^{(2n+1)}_{2j+1}, \qquad 0 \leqslant n \leqslant N-1$$

where $C_l = mh_{l-m}/l$ and $\hat{C}_l = (2N-m)\hat{h}_{2N-m-l}/l$. This immediately implies that the coefficients of the deformation h $(\hat{h})$ can be absorbed into time variables as a shift $t_{2n+1} \to t_{2n+1} + C_{2n+1}$ $(\hat{t}_{2n+1} \to \hat{t}_{2n+1} + \hat{C}_{2n+1})$ so that hodograph solutions $u_j(x, t, \hat{t}; h, \hat{h})$ constructed from deformed cases can be related to those undeformed solutions $u_j(x, t, \hat{t}; h = \hat{h} = 0)$ by shifting the time variables

$$u_j(x, t_n, \hat{t}_n; h, \hat{h}) = u_j(x, t_{2n+1} + C_{2n+1}, \hat{t}_{2n+1} + \hat{C}_{2n+1}; h = \hat{h} = 0).$$

However, if $h(k)$ and $\hat{h}(k)$ are chosen to be odd functions, then it is clear that t^0_{2n+1} in (4.4) vanishes. Let us illustrate hodograph solutions for $N = 1$ and $N = 2$ reductions.

4.1. $N = 1$ reductions

In this case,

$$L = \mathcal{L}^m = \hat{\mathcal{L}}^{m-2} = k^m + muk^{m-2}, \qquad m \in \text{odd}, \tag{4.5}$$

which provide one-variable reductions of the EdBKP system defined by the Lax equations

$$\partial_{t_{2n+1}} L = \{(L^{\frac{2n+1}{m}})_{\geqslant 0}, L\}, \qquad \partial_{\hat{t}_{2n+1}} L = \{(L^{-\frac{2n+1}{m-2}})_{\leqslant -1}, L\}, \qquad n = 0, 1, 2, \ldots,$$

or

$$\partial_{t_{2n+1}} u = \frac{u^n u_x}{n!}\prod_{l=0}^{n}(2n+1-lm),$$

$$\partial_{\hat{t}_{2n+1}} u = -\frac{m(mu)^{\frac{mn+m-1}{2-m}}u_x}{(2-m)^{n+1}n!}\prod_{l=0}^{n}(2n+1-l(2-m)).$$

The solution of the reduced system can be obtained from the hodograph equation (4.1)

$$t_1^0(u) = t_1 + \sum_{j=0}^{\infty} \mu_{2j+1}^{(1)}(u) t_{2j+3} + \sum_{j=0}^{\infty} \hat{\mu}_{2j+1}^{(1)}(u) \hat{t}_{2j+1}, \tag{4.6}$$

where $t_1^0(u)$ is an arbitrary function of u and by (4.2), (4.3) we have

$$\mu_{2j+1}^{(1)}(u) = (2j+3)\mathcal{L}_{[1-m]}^{2j-m+3} = \frac{u^{j+1}}{(j+1)!} \prod_{l=0}^{j+1} (2j+3-lm), \qquad j \geqslant 0,$$

$$\hat{\mu}_{2j+1}^{(1)}(u) = \frac{m(2j+1)}{m-2} \hat{\mathcal{L}}_{[1-m]}^{1-2j-m} = -\frac{m(mu)^{\frac{mj+m-1}{2-m}}}{(2-m)^{j+1} j!} \prod_{l=0}^{j} (2j+1-l(2-m)).$$

Example 1. $N = 1, m = 1$.

In this case, the Lax operator (4.5) has the form

$$\mathcal{L} = \hat{\mathcal{L}}^{-1} = k + uk^{-1},$$

which satisfies the Lax equation

$$\partial_{2n+1}\mathcal{L} = \{(\mathcal{L}^{2n+1})_{\geqslant 0}, \mathcal{L}\}, \qquad \hat{\partial}_{2n+1}\mathcal{L} = \{(\hat{\mathcal{L}}^{-2n-1})_{\leqslant -1}, \mathcal{L}\}, \qquad n = 0, 1, 2, \ldots.$$

Since $\hat{\mathcal{L}}^{-1} = \mathcal{L}$, the coefficient function u depends on t_{2n+1} and $\hat{t}_{2n+1}$ only through the linear combinations $t_{2n+1} - \hat{t}_{2n+1}$. The first three nontrivial equations of them are shown as follows:

$$\partial_{t_3} u = 6uu_x, \qquad \partial_{t_5} u = 30u^2 u_x, \qquad \partial_{t_7} u = 140u^3 u_x, \tag{4.7}$$

$$\partial_{\hat{t}_1} u = -u_x, \qquad \partial_{\hat{t}_3} u = -6uu_x, \qquad \partial_{\hat{t}_5} u = -30u^2 u_x. \tag{4.8}$$

We remark that a solution of the first two equations of (4.7), i.e., t_3- and t_5-flows, also satisfies the dBKP equation (1.7). To find $(2+1)$-dimensional solutions of (4.7), we set $\hat{t}_{2n+1} = 0$ and expand the hodograph equation (4.6) up to $t_5 = t$:

$$t_1^0(u) = x + \mu_1^{(1)} y + \mu_3^{(1)} t = x + 6uy + 30u^2 t. \tag{4.9}$$

Choosing, for example, $t_1^0(u) = u$ and $t_1^0(u) = u^2$ (corresponding to $\mu_1^{(1)}$ and $\mu_3^{(1)}$, respectively) into the hodograph equation (4.9), then we have

$$u(x, y, t) = \frac{1}{60t}(-6y + 1 \pm \sqrt{(6y-1)^2 - 120xt}), \qquad t_1^0(u) = u,$$

$$u(x, y, t) = \frac{1}{30t-1}(-3y \pm \sqrt{9y^2 - (30t-1)x}), \qquad t_1^0(u) = u^2.$$

Furthermore, the $(2+1)$-dimensional solutions include $(x, y, \hat{t}_1)$ in (4.7) and (4.8) can be given by expand the hodograph equation (4.6) up to $t_3 = y$ and $\hat{t}_1$:

$$t_1^0(u) = x + \mu_1^{(1)} y + \hat{\mu}_1^{(1)} \hat{t}_1 = x + 6uy - \hat{t}_1. \tag{4.10}$$

Choosing, $t_1^0(u) = u$ and $t_1^0(u) = u^2$ (corresponding to $\hat{\mu}_3^{(1)}$ and $\hat{\mu}_5^{(1)}$, respectively) into (4.10), we get

$$u(x, y, \hat{t}_1) = -\frac{x - \hat{t}_1}{6y - 1}, \qquad t_1^0(u) = u,$$

$$u(x, y, \hat{t}_1) = 3y \pm \sqrt{9y^2 + x - \hat{t}_1}, \qquad t_1^0(u) = u^2.$$

4.2. $N = 2$ reductions

In this case,

$$L = \mathcal{L}^m = \hat{\mathcal{L}}^{m-4} = k^m + muk^{m-2} + wk^{m-4}, \qquad m \in \text{odd}, \tag{4.11}$$

which describes a class of two-variable Lax reductions of the EdBKP system and satisfies the Lax equations

$$\partial_{t_{2n+1}} L = \left\{\left(L^{\frac{2n+1}{m}}\right)_{\geqslant 0}, L\right\}, \qquad \partial_{\hat{t}_{2n+1}} L = \left\{\left(L^{-\frac{2n+1}{m-4}}\right)_{\leqslant -1}, L\right\}, \qquad n = 0, 1, 2, \ldots,$$

or

$$\begin{aligned}
\frac{\partial u}{\partial t_{2n+1}} &= \sum_{j=0}^{n+1} \binom{\frac{2n+1}{m}}{j}\binom{j}{n-j+1}((mu)^{2j-n-1}w^{n-j+1})_x\\
\frac{\partial w}{\partial t_{2n+1}} &= \sum_{j=0}^{n} \binom{\frac{2n+1}{m}}{j}\binom{j}{n-j}((mu)^{2j-n}w^{n-j}w_x + (4-m)((mu)^{2j-n}w^{n-j})_x w),\\
\frac{\partial u}{\partial \hat{t}_{2n+1}} &= -\sum_{j=0}^{n} \binom{\frac{2n+1}{4-m}}{j}\binom{j}{n-j}((mu)^{2j-n}w^{\frac{2n+1}{4-m}-j})_x,\\
\frac{\partial w}{\partial \hat{t}_{2n+1}} &= -\sum_{j=0}^{n+1} \binom{\frac{2n+1}{4-m}}{j}\binom{j}{n-j+1}\\
&\quad \times \left((mu)^{2j-n-1}w^{\frac{2n+1}{4-m}-j}w_x + (4-m)((mu)^{2j-n-1}w^{\frac{2n+1}{4-m}-j})_x w\right).
\end{aligned} \tag{4.12}$$

By (4.1), the solutions for higher flows of the reduction can be given by the hodograph equations

$$\begin{aligned}
t_1^0(u, w) &= t_1 + \sum_{j=0}^{\infty} \mu_{2j+1}^{(1)}(u, w)t_{2j+5} + \sum_{j=0}^{\infty} \hat{\mu}_{2j+1}^{(1)}(u, w)\hat{t}_{2j+1},\\
t_3^0(u, w) &= t_3 + \sum_{j=0}^{\infty} \mu_{2j+1}^{(3)}(u, w)t_{2j+5} + \sum_{j=0}^{\infty} \hat{\mu}_{2j+1}^{(3)}(u, w)\hat{t}_{2j+1},
\end{aligned} \tag{4.13}$$

where $\mu_{2j+1}^{(i)}(u, w)$ and $\hat{\mu}_{2j+1}^{(i)}(u, w)$ are the functions of u and w, defined by (4.2) and (4.3)

$$\begin{aligned}
\mu_{2j+1}^{(1)} &= (5+2j)\left[\sum_{n=1}^{2+j} \binom{\frac{5+2j-m}{m}}{n}\binom{n}{j-n+2}(mu)^{2n-j-2}w^{2+j-n}\right.\\
&\quad \left. + \frac{m-3}{m}\sum_{n=1}^{1+j} \binom{\frac{5+2j-m}{m}}{n}\binom{n}{j-n+1}(mu)^{2n-j}w^{1+j-n}\right],\\
\mu_{2j+1}^{(3)} &= \frac{5+2j}{3}\sum_{n=1}^{1+j} \binom{\frac{5+2j-m}{m}}{n}\binom{n}{j-n+1}(mu)^{2n-j-1}w^{1+j-n},\\
\hat{\mu}_{2j+1}^{(1)} &= \frac{m(2j+1)}{m-4}\left[\sum_{n=0}^{j-1} \binom{\frac{3-2j-m}{m-4}}{n}\binom{n}{j-n-1}(mu)^{2n-j+1}w^{\frac{3-2j-m}{m-4}-n}\right.\\
&\quad \left. + \frac{m-3}{m}\sum_{n=0}^{j} \binom{\frac{3-2j-m}{m-4}}{n}\binom{n}{j-n}(mu)^{2n-j+1}w^{\frac{3-2j-m}{m-4}-n}\right],\\
\hat{\mu}_{2j+1}^{(3)} &= \frac{m(2j+1)}{3(m-4)}\sum_{n=0}^{j} \binom{\frac{3-2j-m}{m-4}}{n}\binom{n}{j-n}(mu)^{2n-j}w^{\frac{3-2j-m}{m-4}-n}.
\end{aligned} \tag{4.14}$$

Example 2. $N = 2, m = 1$.

In this case, the Lax operator (4.11) is defined by

$$L = \mathcal{L} = \hat{\mathcal{L}}^{-3} = k + uk^{-1} + wk^{-3},$$

which satisfies the Lax equation

$$\partial_{t_{2n+1}}\mathcal{L} = \{(\mathcal{L}^{2n+1})_{\geqslant 0}, \mathcal{L}\}, \qquad \partial_{\hat{t}_{2n+1}}\mathcal{L} = \{(\hat{\mathcal{L}}^{-2n-1})_{\leqslant -1}, \mathcal{L}\}, \qquad n = 0, 1, 2, \ldots.$$

The hierarchy flows of u, w can be read by substituting $m = 1$ into (4.12). From equations (4.14), the coefficients $\mu^{(i)}_{2j+1}(u, w)$ and $\hat{\mu}^{(i)}_{2j+1}(u, w)$ $(i = 1, 3)$ are given by

$$\mu^{(1)}_1 = -10u^2 + 20w, \qquad \mu^{(1)}_3 = -70u^3 + 126uw, \qquad \mu^{(1)}_5 = -378u^4 + 504u^2w + 252w^2,$$
$$\mu^{(3)}_1 = 20u/3, \qquad \mu^{(3)}_3 = 35u^2 + 14w, \qquad \mu^{(3)}_5 = 168u^3 + 168uw,$$
$$\hat{\mu}^{(1)}_1 = \tfrac{2}{3}uw^{-2/3}, \qquad \hat{\mu}^{(1)}_3 = -1, \qquad \hat{\mu}^{(1)}_5 = -\tfrac{10}{27}u^3w^{-4/3} + \tfrac{10}{9}uw^{-1/3},$$
$$\hat{\mu}^{(3)}_1 = -\tfrac{1}{9}w^{-2/3}, \qquad \hat{\mu}^{(3)}_3 = 0, \qquad \hat{\mu}^{(3)}_5 = \tfrac{5}{81}u^2w^{-4/3} - \tfrac{10}{27}w^{-1/3}.$$

To find $(2+1)$-dimensional solutions in $u(x, y, t)$, we set $\hat{t}_{2n+1} = 0$ and expand the hodograph equations (4.13) up to t_5:

$$t^0_1(u, w) = x + \mu^{(1)}_1 t = x + (-10u^2 + 20w)t,$$
$$t^0_3(u, w) = y + \mu^{(3)}_1 t = y + \tfrac{20}{3}ut,$$

where t^0_1 and t^0_3 can be given by $\mu^{(1)}_3$ and $\mu^{(3)}_3$, respectively, as

$$t^0_1(u, w) = 10u^3 - 18uw, \qquad t^0_3(u, w) = -5u^2 - 2w.$$

After eliminating w, we obtain a hodograph equation for u

$$165u^3 + 360tu^2 + 200t^2u + 27yu + 30ty - 3x = 0.$$

The above equation has a real solution

$$u(x, y, t) = \frac{f}{330} + \frac{2(680t^2 - 297y)}{33f} - \frac{8t}{11}$$

with

$$f = (2016\,000t^3 - 1128\,600ty + 326\,700x + 220\sqrt{a_1t^6 + a_2t^4y + a_3t^3x + a_4t^2y^2 + a_5tyx + a_6y^3 + a_7x^2})^{1/3},$$

$$a_1 = 32\,000\,000, \qquad a_2 = -25\,920\,000, \qquad a_3 = 27\,216\,000, \qquad a_4 = -3426\,300,$$
$$a_5 = -1523\,6100, \qquad a_6 = 4330\,260, \qquad a_7 = 2205\,225.$$

One can verify that $u(x, y, t)$ satisfies the t_{2n+1}-flow (4.12) as well as the dBKP equation (1.7). Similarly, the $(2+1)$-dimensional solutions involving $(x, y, \hat{t}_1)$ that satisfy (4.12) can be given by expanding the hodograph equation (4.13) up to $t_3 = y$ and $\hat{t}_1$:

$$t^0_1(u, w) = x + \hat{\mu}^{(1)}_1\hat{t}_1 = x + \tfrac{2}{3}uw^{-2/3}\hat{t}_1, \qquad t^0_3(u, w) = y + \hat{\mu}^{(3)}_1\hat{t}_1 = y - \tfrac{1}{9}w^{-2/3}\hat{t}_1.$$

Choosing $t^0_1 = \frac{9}{5}\hat{\mu}^{(1)}_5$, $t^0_3 = \frac{9}{5}\hat{\mu}^{(3)}_5$, after eliminating w, we get an implicit equation,

$$1296y^4u^4 + 864xy^3u^3 + 72y(3x^2y + 2y\hat{t}_1 + 2)u^2 + 24x(x^2y + 2y\hat{t}_1 + 2)u + x^2(x^2 + 4\hat{t}_1) = 0,$$

which can be solved as

$$u(x, y, \hat{t}_1) = -\frac{x}{6y} \pm \frac{1}{6y^3}\sqrt{2y^3(-y\hat{t}_1 - 1 \pm \sqrt{x^2y + (y\hat{t}_1 + 1)^2}}).$$

So far, we have obtained some hodograph solutions of the $(2+1)$-dimensional dBKP equation (and its extension). To get more new solutions in this approach, the main difficulty we have to confront with is to solve higher order algebraic equations. In the next section, we shall show that Bäcklund transformations provide a convenient way to construct new solutions by treating the known solutions as seed solutions.

5. Bäcklund transformations

Let us discuss the one-parameter flow generated by addition symmetries. Setting $s = -\epsilon$, then the additional flow (3.3) can be written as

$$\frac{\partial \mathcal{L}}{\partial s} = \{\mathcal{L}, (F(\mathcal{L}, \mathcal{M}) - \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}))_{\leqslant -1}\}, \tag{5.1}$$

where the generators F and $\hat{F}$ can be expressed as

$$F(\mathcal{L}, \mathcal{M}) = \sum_{ij} c_{ij}\mathcal{L}^{2i+1}\mathcal{M}^j, \qquad \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) = \sum_{ij} \hat{c}_{ij}\hat{\mathcal{L}}^{2i+1}\hat{\mathcal{M}}^j$$

with c_{ij} and $\hat{c}_{ij}$ being the arbitrary constants. Motivated by the dKP case [19], let us consider an (r, s)-restricted system by putting $t_{2n+1} = 0, \forall n \geqslant r+1$, and $\hat{t}_{2n+1} = 0, \forall n \geqslant s$. The Orlov–Schulman operators then become

$$\mathcal{M} = (2r+1)t_{2r+1}\mathcal{L}^{2r} + (2r-1)t_{2r-1}\mathcal{L}^{2r-2} + \cdots + x + O(\mathcal{L}^{-2}),$$
$$\hat{\mathcal{M}} = -(2s-1)\hat{t}_{2s-1}\hat{\mathcal{L}}^{-2s} - (2s-3)\hat{t}_{2s-3}\hat{\mathcal{L}}^{-2s+2} + \cdots - \hat{t}_1\hat{\mathcal{L}}^{-2} + \hat{v}_2 + O(\hat{\mathcal{L}}^{2}).$$

Substituting $\mathcal{M}$ and $\hat{\mathcal{M}}$ into F and $\hat{F}$, respectively, and demanding that additional flows (5.1) generated by F and $\hat{F}$ do not induce hierarchy flows for $t_{2n+1}(n \geqslant r+1)$ and $\hat{t}_{2n+1}(n \geqslant s)$, then we have

$$F(\mathcal{L}, \mathcal{M}) = \sum_{i=0}^{r} F_i(\mathcal{L}, \mathcal{M}), \qquad \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) = \sum_{i=-s}^{-1} \hat{F}_i(\hat{\mathcal{L}}, \hat{\mathcal{M}})$$

where the $r+s+1$ symmetry generators are defined by

$$F_i(\mathcal{L}, \mathcal{M}) = \alpha_i\left(\frac{\mathcal{M}}{(2r+1)\mathcal{L}^{2r}}\right)\mathcal{L}^{2i+1}, \qquad i = 0, 1, 2, \ldots, r (r \geqslant 1),$$
$$\hat{F}_i(\hat{\mathcal{L}}, \hat{\mathcal{M}}) = \beta_i\left(-\frac{\hat{\mathcal{M}}\hat{\mathcal{L}}^{2s}}{(2s-1)}\right)\hat{\mathcal{L}}^{2i+1}, \qquad i = -s, -s+1, \ldots, -1.$$

and α_i, β_i are the arbitrary functions.

Let us discuss the simplest nontrivial example, the (2, 1)-restricted system, which depends on four time variables $\hat{t}_1$, x, y and t. The four symmetry generators in the system are given by

$$\hat{F}_{-1} = \beta_{-1}(-\hat{\mathcal{M}}\hat{\mathcal{L}}^2)\hat{\mathcal{L}}^{-1}, \qquad F_0 = \alpha_0\left(\frac{\mathcal{M}}{5\mathcal{L}^4}\right)\mathcal{L},$$
$$F_1 = \alpha_1\left(\frac{\mathcal{M}}{5\mathcal{L}^4}\right)\mathcal{L}^3, \qquad F_2 = \alpha_2\left(\frac{\mathcal{M}}{5\mathcal{L}^4}\right)\mathcal{L}^5$$

where

$$\beta_{-1}(-\hat{\mathcal{M}}\hat{\mathcal{L}}^2) = \beta_{-1}(\hat{t}_1) + O(\hat{\mathcal{L}}^2)$$
$$\alpha_i\left(\frac{\mathcal{M}}{5\mathcal{L}^4}\right) = \alpha_i(t_5) + \frac{3}{5}\alpha_i'(t_5)t_3\mathcal{L}^{-2} + \left(\frac{1}{5}\alpha_i'(t_5)x + \frac{9}{50}\alpha_i''(t_5)t_3^2\right)\mathcal{L}^{-4}$$
$$+ \left(\frac{1}{5}\alpha_i'(t_5)v_2 + \frac{3}{25}\alpha_i''(t_5)xt_3 + \frac{9}{250}\alpha_i'''t_3^3\right)\mathcal{L}^{-6} + O(\hat{\mathcal{L}}^{-8}), \qquad i = 0, 1, 2.$$

The additional flows for the primary variable $u \equiv u_2$ are

$$\hat{F}_{-1}: \quad \frac{\partial u}{\partial s} = \beta_{-1}(\hat{t}_1)\frac{\partial u}{\partial \hat{t}_1},$$

$$F_0: \quad \frac{\partial u}{\partial s} = \alpha_0(t_5)\frac{\partial u}{\partial x},$$

$$F_1: \quad \frac{\partial u}{\partial s} = \alpha_1(t_5)\frac{\partial u}{\partial t_3} + \frac{3}{5}\alpha_1'(t_5)t_3\frac{\partial u}{\partial x} + \frac{1}{5}\alpha_1'(t_5),$$

$$F_2: \quad \frac{\partial u}{\partial s} = \alpha_2(t_5)\frac{\partial u}{\partial t_5} + \frac{3}{5}\alpha_2'(t_5)t_3\frac{\partial u}{\partial t_3} + \left(\frac{1}{5}\alpha_2'(t_5)x + \frac{9}{50}\alpha_2''(t_5)t_3^2\right)\frac{\partial u}{\partial x} + \frac{2}{5}u\alpha_2'(t_5) + \frac{3}{25}\alpha_2''(t_5)t_3,$$

where we have used the fact $v_{2x} = u$ in the last equation. These additional flows can be solved as follows:

$$\hat{F}_{-1}: \quad u(s; x, y, t, \hat{t}_1) = u(x, y, t, \tilde{\hat{t}}_1),$$

$$F_0: \quad u(s; x, y, t, \hat{t}_1) = u(x + s\alpha_0(t), y, t, \hat{t}_1),$$

$$F_1: \quad u(s; x, y, t, \hat{t}_1) = u\big(x + \tfrac{3}{10}s^2\alpha_1(t)\alpha_1'(t) + \tfrac{3}{5}sy\alpha_1'(t), y + s\alpha_1(t), t, \hat{t}_1\big) + \tfrac{1}{5}s\alpha_1'(t),$$

$$F_2: \quad u(s; x, y, t, \hat{t}_1) = (\tilde{t}')^{2/5}u\big((\tilde{t}')^{1/5}\big(x + \tfrac{9}{50}(\tilde{t}')^{-1}\tilde{t}''y^2\big), (\tilde{t}')^{3/5}y, \tilde{t}, \hat{t}_1\big) + \tfrac{3}{25}(\tilde{t}')^{-1}\tilde{t}''y,$$

where $\tilde{\hat{t}}_1(s, \hat{t}_1)$ and $\tilde{t}(s, t)$ are defined implicitly by

$$\int^{\tilde{\hat{t}}_1(s,\hat{t}_1)} \frac{\mathrm{d}z}{\beta_{-1}(z)} = s + \int^{\hat{t}_1} \frac{\mathrm{d}z}{\beta_{-1}(z)}, \qquad \int^{\tilde{t}(s,t)} \frac{\mathrm{d}z}{\alpha_2(z)} = s + \int^{t} \frac{\mathrm{d}z}{\alpha_2(z)}, \tag{5.2}$$

and $\tilde{t}' = \partial\tilde{t}/\partial t$. Since the formulae shown above connect two solutions of the dBKP equation, they can be viewed as Bäcklund transformations of the system. In particular, a composition of any two Bäcklund transformations is still a Bäcklund transformation. Due to the commutative subalgebra of additional symmetries, a pair of F-type and $\hat{F}$-type Bäcklund transformations satisfies the following commutative diagram of transformations:

$$\begin{array}{ccccc} & & u(s; x, y, t, \hat{t}_1) & & \\ & F \nearrow & & \searrow \hat{F} & \\ u(x, y, t, \hat{t}_1) & & & & u(s, \hat{s}; x, y, t, \hat{t}_1) = u(\hat{s}, s; x, y, t, \hat{t}_1) \\ & \hat{F} \searrow & & \nearrow F & \\ & & u(\hat{s}; x, y, t, \hat{t}_1) & & \end{array}$$

which reveals the permutability of Bäcklund transformations.

Example 3. Consider the Bäcklund transformation generated by F_2. Set $\hat{t}_1 = 0$, we have

$$u(s; x, y, t) = (\tilde{t}'(s,t))^{2/5}u((\tilde{t}'(s,t))^{1/5}(x + \tfrac{9}{50}(\tilde{t}'(s,t))^{-1}\tilde{t}''(s,t)y^2), (\tilde{t}'(s,t))^{3/5}y, \tilde{t}(s,t)) + \tfrac{3}{25}(\tilde{t}'(s,t))^{-1}\tilde{t}''(s,t)y. \tag{5.3}$$

Given a hodograph solution of the (2 + 1)-dimensional dBKP equation

$$u(x, y, t) = \frac{1}{60t}(-6y + 1 + \sqrt{(6y-1)^2 - 120xt})$$

which, after the Bäcklund transformation (5.3), becomes

$$u(s; x, y, t) = \frac{(f'(s,t))^{2/5}}{60 f(s,t)}(-6(f'(s,t))^{3/5}y + 1$$
$$+ \sqrt{(6(f'(s,t))^{3/5}y - 1)^2 - 120(f'(s,t))^{1/5}\left(x + \frac{9}{50}(f'(s,t))^{-1}f''(s,t)y^2\right)f(s,t)})$$
$$+ \frac{3}{25}(f'(s,t))^{-1}f''(s,t)y,$$

where $f(s,t) \equiv \tilde{t}(s,t)$ defined by (5.2) is an arbitrary well-behaved function with respect to t. Therefore, we obtain a one-parameter family of solutions of the $(2+1)$-dimensional dBKP equation (1.5). Other transformations can be performed in a similar manner so we omit them here.

6. Miura transformation

Let us perform a Miura transformation to the dBKP hierarchy (the t_{2n+1}-flow part of (2.1) and (2.4) only):

$$\mathcal{L}' = \mathrm{e}^{-\mathrm{ad}\phi}(\mathcal{L}), \qquad \mathcal{M}' = \mathrm{e}^{-\mathrm{ad}\phi}(\mathcal{M}),$$

where $\phi = \phi(t)$ satisfies the evolution equations

$$\partial_{2n+1}\phi = \mathcal{L}_+^{2n+1}|_{k=\phi_x}.$$

Then, the corresponding modified hierarchy is given by

$$\frac{\partial \mathcal{L}'}{\partial t_{2n+1}} = \{\mathcal{B}'_{2n+1}, \mathcal{L}'\}, \qquad \frac{\partial \mathcal{M}'}{\partial t_{2n+1}} = \{\mathcal{B}'_{2n+1}, \mathcal{M}'\},$$

with the canonical Poisson relation

$$\{\mathcal{L}', \mathcal{M}'\} = 1,$$

where $\mathcal{B}'_{2n+1} = (\mathcal{L}'^{2n+1})_{\geqslant 1}$. The new Lax and Orlov–Schulman operators have the form

$$\mathcal{L}' = k + \sum_{n=1}^{\infty} u'_n k^{-n+1}, \qquad \mathcal{M}' = \sum_{n=0}^{\infty}(2n+1)t_{2n+1}\mathcal{L}'^{2n} + \sum_{n=0} v_{2n+2}\mathcal{L}'^{-2n-2},$$

where the coefficients u'_j in $\mathcal{L}'$ are related to those of u_j in $\mathcal{L}$ as

$$u'_1 = \phi_x, \qquad u'_j = u_j + \sum_{n=1}^{j-2}(-\phi_x)^n u_{j-n}\binom{j-2}{n}, \qquad j \geqslant 2$$

with the proviso $u_{2j+1} = 0$. In fact, the Lax flows for $\mathcal{L}'$ describe a modified partner of the dBKP hierarchy, we may call it the dispersionless modified BKP (dmBKP) hierarchy. Consider the zero curvature equations:

$$\frac{\partial \mathcal{B}'_{2n+1}}{\partial t_{2m+1}} - \frac{\partial \mathcal{B}'_{2m+1}}{\partial t_{2n+1}} + \{\mathcal{B}'_{2n+1}, \mathcal{B}'_{2m+1}\} = 0.$$

For $n = 1$ and $m = 2$, we have

$$\frac{\partial \mathcal{B}'_3}{\partial t_5} - \frac{\partial \mathcal{B}'_5}{\partial t_3} + \{\mathcal{B}'_3, \mathcal{B}'_5\} = 0. \tag{6.1}$$

From the coefficient of k^4 in (6.1) we get

$$u'_3 = \tfrac{1}{3}\partial_x^{-1}u'_{1y} - 2u'_1u'_2 - \tfrac{1}{3}u'^3_1,$$

while for that of k^3, after eliminating u_3', we have

$$u_4' = \tfrac{1}{3}\partial_x^{-1}u_{2y}' - u_2'^2 + \tfrac{2}{3}u_1'^4 + 3u_1'^2u_2' - \tfrac{2}{3}u_1'\partial_x^{-1}u_{1y}'.$$

Taking the coefficient of k^2 in (6.1) and eliminating u_3' and u_4' from above, we obtain

$$3u_{1t}' + \left(2u_1'^5 + 10u_1'^3u_2' + 15u_1'u_2'^2 - 5u_2'\partial_x^{-1}u_{1y}'\right)_x \\ - \tfrac{5}{3}\partial_x^{-1}u_{1yy}' - 10u_1'u_{1x}'\partial_x^{-1}u_{1y}' - 5u_{1x}'\partial_x^{-1}u_{2y}' - 5u_{2x}'\partial_x^{-1}u_{1y}' = 0. \tag{6.2}$$

To eliminate u_2', we consider the t_3-flow of the gauge function ϕ:

$$\frac{\partial\phi}{\partial t_3} = \phi_x^3 + 3u_2\phi_x.$$

Since $u_1' = \phi_x$ and $u_2' = u_2$, we obtain the relation between u_1' and u_2' as

$$3u_1'u_2' = -u_1'^3 + \partial_x^{-1}u_{1y}'. \tag{6.3}$$

After substituting (6.3) into (6.2) to eliminate u_2' we get the (2 + 1)-dimensional dmBKP equation ($v = u_1'$):

$$9v_t + 5v^4v_x + 5v^2v_y - 5\partial_x^{-1}v_{yy} - 10vv_x\partial_x^{-1}v_y - 5v^{-1}v_y\partial_x^{-1}v_y \\ + 5v^{-2}v_x\left(\partial_x^{-1}v_y\right)^2 + 5v_x\partial_x^{-1}\left(v^2 - v^{-1}\partial_x^{-1}v_y\right)_y = 0.$$

7. Conclusion

To sum up, we have studied the dBKP hierarchy from its extension, the so-called EdBKP hierarchy which is an integrable hierarchy underlying the Landau–Ginzburg models of D-type proposed by Takasaki. After introducing a dressing formulation to the EdBKP hierarchy, we discuss additional symmetries of its solution space via Riemann–Hilbert problem. Particularly, we discussed finite-dimensional reductions and hodograph solutions of the EdBKP hierarchy and constructed new solutions using Bäcklund transformations generated by additional symmetries. Furthermore, the gauge equivalence to the dBKP hierarchy, the so-called dmBKP hierarchy, is also discussed. Just like the relationship between the dKP and dmKP hierarchies [7], the integrability of the dmBKP can be investigated from that of the dBKP via the Miura transformation between them. Finally, we would like to remark that the Lax as well as Hamiltonian formulations to dispersionless integrable systems can be defined with respect to the Poisson bracket of the form $\{A, B\}^{(r)} = p^r\partial A/\partial k\partial B/\partial x - p^r\partial A/\partial x\partial B/\partial k$ [4, 5, 17, 18]. It would be interesting to discuss the integrability of the dBKP hierarchy in this more general setting.

Acknowledgments

This work is partially supported by the National Science Council of Taiwan under grant no NSC94-2112-M-194-009(MHT).

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